*J. Fluid Mech.* (1973), *vol.* 57, *part* 1, *pp.* 1–25

*Printed in Great Britain*

# A higher order theory for compressible turbulent boundary layers at moderately large Reynolds number

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(Received 9 September 1971 and in revised form 16 June 1972)

A higher order theory for two-dimensional turbulent boundary-layer flow of a compressible fluid past a plane wall is formulated, for moderately large values of the Reynolds number, by the method of matched asymptotic expansions. The parameters $(\gamma-1)M_\infty^2$ and the molecular Prandtl number are assumed to be of order unity. The analysis deals with the set of Reynolds equations of mean motion (which are underdetermined without an additional set of closure hypotheses) and assumes that the non-dimensional fluctuations in velocity, temperature and density are of order $U_*$ (friction velocity divided by free-stream velocity at some designation point), while fluctuations in pressure are of order $U_*^2$. The first-order results of the present study lead to asymptotic laws for velocity and temperature distributions which correspond to the law of the wall, logarithmic law and defect law, and also to skin friction and heat-transfer laws. It turns out that the first-order defect law depends upon the gradient of entropy and stagnation enthalpy and the law of the wall is independent of viscous dissipation. The second-order terms of the present work (accounting for mean convection due to turbulent mass flux, viscous dissipation in the inner flow and displacement effects in the outer flow) describe the necessary corrections to first-order terms due to low Reynolds number effects. In the overlap region the second-order results, for the law of the wall and the defect law, show bilogarithmic terms along with logarithmic terms.

## 1. Introduction

The problem of the turbulent boundary layer has attracted wide attention. It is well known that the study of turbulent boundary layers, indeed that of all turbulent flows, is handicapped by the problem of closure. Despite numerous attempts, a closure hypothesis which describes the essential physics in a reasonably general fashion has yet to be constructed (Lumley 1970) and therefore analyses based upon closure hypotheses which are not fully satisfactory are bound to be subject to some uncertainty. Much of what is known about incompressible turbulent boundary layers stems from experiments. Using dimensional and similarity arguments, general empirical correlations – like the law of the wall, logarithmic law in the overlap region, velocity defect law, law of the wake and skin friction law (see Rotta 1962) – are obtained. However, for the temperature profile in incompressible flow very few measurements have been made, compared with the number of measurements available for velocity profiles (similar comments apply to empirical correlations for the temperature profile); most authors have studied the relation of heat transfer to skin friction and a good review is available in Schlichting (1968). For a compressible flow, too, very few measurements have been made when compared with the number for incompressible flow. As a result no systematic attempts have been made to find general empirical correlations.

These classical empirical laws (law of the wall, etc.), however, are necessarily approximate and small systematic departures are known in some cases. For example, for incompressible boundary-layer flow Rotta (1962, p. 101) finds a higher order term in the velocity defect law and skin friction law. Recent measurements of Simpson (1970) at lower Reynolds number also exhibit such higher order effects. These higher order effects seems to be of the order of the non-dimensional friction velocity $U_* = u_\tau/U_\infty$ (here $u_\tau = (\tau_w/\rho_w)^{\frac{1}{2}}$ is the friction velocity, $\tau_w$ the local shear stress at the wall, $\rho_w$ the local density at the wall and $U_\infty$ the free-stream velocity). For a compressible flow, too, Rotta (1960) has shown that the law of the wall depends upon the *so-called* friction Mach number $M_\tau$. It is shown later that, for a given free-stream Mach number $M_\infty$, the friction Mach number $M_\tau$ is of order $U_*$. Furthermore, the measurements of Lobb, Winkler & Persh (quoted by Schlichting 1968, figure 23.11) and the inspection of data from various sources by Bradshaw & Ferriss (1971) show that the dependence of the law of wall on $M_\tau$ is weak up to an $M_\infty$ of 5. In view of the fact that $U_* \to 0$ as $R \to \infty$ it is preferable to regard the classical empirical laws as asymptotic laws in the sense that these become exact as the Reynolds number approaches infinity. The departure from these laws at lower Reynolds number are the higher order effects. The main aim of the present work is to formulate a higher order theory for compressible turbulent boundary-layer flow of a perfect gas with constant specific heats when $(\gamma-1)M_\infty^2$ and molecular Prandtl number are of order unity. The higher order analysis for an incompressible flow is described in the appendix. For laminar flow, such a higher order theory has been formulated by Van Dyke (1962*a*, *b*).

It is well known that a satisfactory prediction about a turbulent boundary layer cannot, in general, be made by any of the older theories relying on such concepts as mixing length or eddy viscosity (Bradshaw 1968; Phillips 1969; Kline, Moffatt & Morkovin 1969). For incompressible turbulent boundary layers, rather more elaborate methods have recently been devised, and are based either on further physical assumptions (Head 1958; Narasimha 1969) or hypotheses on the turbulent energy equation (Glushko 1965; Bradshaw, Ferriss & Atwell 1967; Donaldson & Rosenbaum 1969) or deal with the underdetermined system of equations of mean motion, i.e. without an additional set of closure hypothesis (Millikan 1939; Gill 1968; Yajnik 1970; Afzal & Yajnik 1971, 1972). Various authors (Glushko 1965; Bradshaw *et al.* 1967, etc.), in addition to the usual equation of mean motion, have employed the equation for transport of turbulent kinetic energy. To make the system of equations closed, the terms in the turbulent kinetic energy equation are modelled by making five hypotheses, and each of these authors has used quite different hypotheses (see Donaldson & Rosenbaum 1969). The analyses of Yajnik (1970) and Afzal & Yajnik (1971) deal with the undetermined set of equations of mean motion and provide results to lowest order. This approach is somewhat similar to that of Millikan (1939) and Gill (1968), who have established relationships among the empirical correlations without invoking a hypothetical model of turbulence. Generalizations of the schemes which deal with the underdetermined set of equations of mean motion to transpiration, higher orders and compressible flows are needed. The present work deals with the problem of higher order effects in compressible turbulent boundary layers, while for the problem of transpiration we wish to make some comments. The problem of transpiration ($V_w$, the normal velocity at the wall, is of the order of the friction velocity $U_*$) needs, in addition to inner and outer limits (in the terminology of Yajnik), an inviscid limit with corresponding inviscid expansions, so as to account for the effects of displacement of the lowest order inviscid solution by the outer solution. This displacement gives rise to the displacement speed and to induced pressure, both of order $U_*$. This is because on the solid surface $d\delta_1/dx$ is of $O(U_*^2)$ but with transpiration it is of $O(V_w/U_\infty)$. Thus with transpiration, even the lowest order problem (of order $U_*$) is global in nature. None of the earlier studies have noticed this displacement effect and all these authors have treated the problem as a local one (see Jeromin 1969). Furthermore, in the presence of transpiration the inner expansions and the corresponding order hypotheses also need some changes. Further, the evaluation of higher order terms (without transpiration) also calls for the introduction of an inviscid limit with corresponding inviscid expansions.

The problem of compressible turbulent boundary layers has been studied by Herring & Mellor (1969) and Cebeci & Smith (1970), using a set of closure hypotheses which rely on local equilibrium relations between the gradients of mean quantities and turbulent (Reynolds) terms (eddy viscosity and eddy conductivity) and ignoring the effects of turbulent history. The method of Head (1958) is extended by Green (1968) to compressible flows. Bradshaw & Ferriss (1971) have extended their earlier analysis of incompressible flow to the case of compressible turbulent boundary layers with adiabatic walls. In their analysis the latter authors have simplified the equation of mean momentum and turbulent kinetic energy through the so-called Morkovin (1964) hypothesis, supported by the experiments of Kistler (1959) and Demetriades (1968), which says that the structure of turbulence (specifically, dimensionless quantities like asnisotropy parameters, spectrum shapes and the like) will not be affected by compressibility as long as $M_\infty$ is less than 5.

The main aim of the present work is to formulate a general theory for higher order effects in turbulent boundary-layer flow of a compressible fluid past a plane body, at large Reynolds number, by the method of matched asymptotic expansions. The on-coming stream need not be iso-energetic and can have gradients of entropy and stagnation enthalpy. The parameters $(\gamma-1)M_\infty^2$ and $\sigma$ (the molecular Prandtl number) are assumed to be of order unity. The assumption $(\gamma-1)M_\infty^2 = O(1)$ probably restricts $M_\infty$ to a maximum of 5. For this case

($M_\infty < 5$) it is plausible to assume (see § 3) that the dimensionless fluctuations in velocity, temperature and density are of equal order, say, $E$ (later shown to be $U_*$ at some designated point on the wall), and fluctuations in pressure are of the order of the local mean shear.

The present analysis deals with the underdetermined set of equations of mean motion (without an additional set of closure hypotheses) and assumes that fluctuations in velocity, temperature and density are of order $U_*$ and the fluctuations in pressure are of order $U_*^2$. It may be noted that these assumptions do not impose (*a*) any condition (like Morkovin's hypothesis) regarding the effects of compressibility or (*b*) any functional relationship between the quantities determined by mean motion and those depending on the turbulence. As a result, these assumptions are not the same as closure hypotheses. In the present work the first-order results depend upon gradients of entropy and stagnation enthalpy in the oncoming stream and turn out to be independent of viscous dissipation. The first-order analysis shows asymptotic laws for velocity, temperature and density distributions which correspond to the law of the wall, logarithmic law and defect law, and also skin friction and heat-transfer laws. The second-order terms of the present work describe necessary corrections to these laws at lower Reynolds numbers and account for viscous dissipation and mean convection in the inner region and displacement effects. In the overlap region of the law of the wall and the defect law, the second-order corrections show bilogarithmic and logarithmic terms.

At this stage, it is instructive to examine Rotta's (1960) analysis of a compressible turbulent boundary layer with heat transfer. In his analysis Rotta (1960) has patched the law of the wall to the velocity defect law at the so-called point of maximum temperature, rather than matching them asymptotically. This point of maximum temperature, according to Rotta, is located deep within the sublayer, and therefore depends upon the conditions at the wall (inner variables), in particular, the viscosity. The law of the wall proposed by Rotta is

$$\bar{u} = u_\tau f(\bar{y}u_\tau/\bar{\nu}_w, T_*, M_\tau, \gamma, \sigma, \omega). \tag{1}$$

Here $\bar{u}$ and $\bar{y}$ are the dimensional tangential velocity and normal co-ordinate. $T_* = -Q_w/(\rho_w C_p u_\tau T_w)$ is a dimensionless heat-flux parameter which we shall call the (dimensionless) friction temperature, $Q_w$ is the local heat flux at the wall, $u_\tau$ is the friction velocity $(\tau_w/\rho_w)^{\frac{1}{2}}$, $T_w$ is the wall temperature, $\bar{\nu}_w$ is the local dimensional kinematic viscosity, $\gamma$ the ratio of specific heats and $\omega$ is the index of the power law for the viscosity–temperature relationship. The characteristic parameter for the compressibility is the friction Mach number $M_\tau$ ($= u_\tau/a_w$, where $a_w$ is the sonic velocity at wall temperature). For a given free-stream Mach number $M_\infty$, the friction Mach number $M_\tau$ is of the order of the dimensionless friction velocity $U_*$ (Rotta 1960). Thus for a fixed $M_\infty$, as $R$ approaches infinity $M_\tau$ approaches zero like $U_*$.

As will be shown in § 3, natural choices of scales for the velocity and temperature fluctuations are $U_*$ and $T_*$ respectively, and $T_*$ is of order $U_*$. Without loss of generality, let

$$T_* = A_t U_*, \tag{2}$$

where the quantity $A_t$ is of order unity. Now using the above arguments an asymptotic expansion of (1), due to Rotta (1960), as $R \to \infty$ for fixed $\bar{y}u_\tau/\bar{\nu}_w$ is

$$\frac{\bar{u}}{U_\infty} = \frac{u_\tau}{U_\infty} f_1(\bar{y}u_\tau/\bar{\nu}_w, \sigma, \gamma, \omega) + \left(\frac{u_\tau}{U_\infty}\right)^2 f_2(\bar{y}u_\tau/\bar{\nu}_w, M_\infty, A_t, \sigma, \gamma, \omega) + \dots \quad (3)$$

This expression (3) as well as the present work shows that the first-order terms are independent of $M_\infty$ and $A_t$.

## 2. Governing equations of mean motion

The governing equations of mean motion for two-dimensional turbulent flow of a compressible fluid in non-dimensional form (all lengths are non-dimensionalized by a typical body dimension $L$, speeds by the characteristic reference speed $U_\infty$, pressure by $\rho_\infty U_\infty^2$, temperature by $T_\infty$, density by $\rho_\infty$, viscosity and second viscosity by $\mu_\infty$) are as follows.

Continuity equation:

$$(\rho U + \overline{\rho' u'})_x + (\rho V + \overline{\rho' v'})_y = 0. \quad (4)$$

Momentum equations:

$$\begin{aligned}(\rho U + \overline{\rho' u'}) U_x + (\rho V + \overline{\rho' v'}) U_y + P_x \\ = R^{-1} L_x(\mu, \lambda, U, V) + R^{-1} \overline{L_x(\mu', \lambda'; u', v')} \\ - (\rho\overline{u'u'} + U\overline{\rho' u'} + \overline{\rho' u' u'})_x - (\rho\overline{u'v'} + V\overline{\rho' u'} + \overline{\rho' u' v'})_y,\end{aligned} \quad (5)$$

$$\begin{aligned}(\rho U + \overline{\rho' u'}) V_x + (\rho V + \overline{\rho' v'}) V_y + P_y \\ = R^{-1} L_y(\mu, \lambda; U, V) + R^{-1} \overline{L_y(\mu', \lambda'; u', v')} \\ - (\rho\overline{u'v'} + V\overline{\rho' u'} + \overline{\rho' u' v'})_x - (\rho\overline{v'v'} + V\overline{\rho' v'} + \overline{\rho' v' v'})_y,\end{aligned} \quad (6)$$

where $L_x$ and $L_y$ are used to denote the viscous terms in the $x$ and $y$ momentum equations and are given by

$$L_x(\mu, \lambda; U, V) = (\mu U_y + \mu V_x)_y + 2(\mu U_x)_x + (\lambda U_x + \lambda V_y)_x, \quad (7a)$$

$$L_y(\mu, \lambda; U, V) = (\mu U_y + \mu V_x)_x + 2(\mu V_y)_y + (\lambda U_x + \lambda V_y)_y. \quad (7b)$$

Energy equation:

$$\begin{aligned}(\rho U + \overline{\rho' u'}) T_x + (\rho V + \overline{\rho' v'}) T_y - D(UP_x + VP_y) \\ = (R\sigma)^{-1} [L_t(\mu, T) + \overline{L_t(\mu', t')} - (\rho\overline{u't'} + U\overline{\rho' t'} + \overline{\rho' t' u'})_x \\ - (\rho\overline{v't'} + V\overline{\rho' t'} + \overline{\rho' t' v'})_y + D[(\overline{u'p'})_x + (\overline{v'p'})_y - \overline{p'(u'_x + v'_y)}] \\ + DR^{-1}[\Phi + \phi'].\end{aligned} \quad (8)$$

Here

$$L_t(\mu, T) = (\mu T_x)_x + (\mu T_y)_y \quad (9a)$$

denotes the conduction terms, $\Phi$ is the mean dissipation defined by

$$\Phi = \mu[2U_x^2 + 2V_y^2 + (U_y + V_x)^2] + \lambda(U_x + V_y)^2, \quad (9b)$$

the turbulent dissipation $\phi'$ is given by

$$\begin{aligned}\phi' = \mu[2\overline{u_x'^2} + 2\overline{v_x'^2} + \overline{(u'_y + v'_x)^2}] + \lambda\overline{(u'_x + v'_y)^2} + 2\overline{\mu'(2U_x u'_x + u_x'^2)} \\ + \overline{2\mu'(2V_y v'_y + v_y'^2)} + \overline{\mu'(u'_y + v'_x)^2} + 2(U_y + V_x)\overline{\mu'(u'_y + v'_x)} + \overline{\lambda'(u'_x + v'_y)^2} \\ + 2(U_x + V_y)\overline{\lambda'(u'_x + v'_y)}\end{aligned} \quad (9c)$$

and $D$ is the compressibility factor defined by

$$D = (\gamma - 1)\, M_\infty^2. \tag{9d}$$

Equations of state:

$$P = [(\gamma - 1)/\gamma]\,(\rho T + \overline{\rho' t'})/D, \tag{10a}$$

$$p' = [(\gamma - 1)/\gamma]\,(\rho' T + t'\rho + t'\rho' - \overline{t'\rho'})/D. \tag{10b}$$

Equation (10*b*) for the pressure fluctuation $p'$ is obtained by subtracting (10*a*) from the instantaneous equation of state.

Here $(U, V)$, $P$, $\rho$, $T$, $\mu$ and $\lambda$ are the non-dimensional mean velocity components, mean pressure, mean density, mean temperature, mean viscosity and mean second viscosity respectively. The non-dimensional fluctuations of these quantities are denoted by $(u', v')$, $p'$, $\rho'$, $t'$, $\mu'$ and $\lambda'$. The non-dimensional coordinate along the body is $x$ and normal to it is $y$. The suffix $x$ or $y$ denotes partial differentiation with respect to $x$ or $y$. The quantity $R = U_\infty \rho_\infty L/\mu_\infty$ is the characteristic Reynolds number, $\sigma = \mu C_p/K$ is the molecular Prandtl number, $M_\infty$ is a characteristic Mach number and $\gamma$ is the ratio of specific heats.

The boundary conditions to be satisfied by the mean flow at the solid surface are

$$y = 0, \quad U = V = 0, \quad T = T_w(x), \tag{11}$$

and the turbulent fluctuations must vanish at the wall. Upstream the flow has to approach prescribed (possibly) non-uniform velocity and temperature fields.

## 3. Scales for turbulent fluctuations and asymptotic expansions

In this section we first analyse the turbulent fluctuations (non-dimensionalized as in §2) of velocity, pressure, temperature and density and estimate their relative orders of magnitude. The present work deals with the case where $(\gamma - 1)\, M_\infty^2$ is of order unity, which probably restricts the upper limit of $M_\infty$ to 5 (non-hypersonic flows). For $M_\infty$ below 5, experimental results concerning the fluctuating flow field are available and help us to estimate the orders of magnitude of the fluctuations under consideration. These experiments are analysed below.

### 3.1. *Velocity fluctuations*

The measurements of Kistler (1959) up to a Mach number of 4·62 show that the distribution of the r.m.s. turbulent velocity fluctuation $(\overline{u'^2})^{\frac{1}{2}}/U_*$ is qualitatively similar to that found in the incompressible case (see Kistler 1959, figure 10, p. 294), suggesting that the whole turbulence production mechanism is similar to that of an incompressible flow in spite of the presence of temperature fluctuations (Laufer 1969). This observation leads to the assumption that the fluctuation $u'$ is of the order of the friction velocity, i.e.

$$u' \sim U_*. \tag{12a}$$

### 3.2. *Static temperature fluctuations*

According to the measurements of Kistler (1959, p. 296, figure 11) the static temperature fluctuation scales with an average mean static temperature across the boundary layer, i.e.

$$\frac{(\overline{t'^2})^{\frac{1}{2}}}{T} = 2\frac{T_0 - T_\infty}{T_0 + T_\infty} f\left(\frac{y}{\delta}\right),$$

where $f(y/\delta)$ is a function independent of Mach number $M_\infty$ having a maximum of about 0·1 (Laufer 1969). When $T_0 - T_\infty$ is of order unity (as in the present work) the fluctuation in temperature scales with the mean static temperature $T$. The appropriate scale for the mean static temperature is the friction temperature $T_*$ (see Bradshaw & Ferriss 1970; Rotta 1960, 1964). This suggests that the scale for static temperature fluctuation $t'$ is the friction temperature $T_*$, i.e.

$$t' \sim T_*. \tag{12b}$$

### 3.3. *Static pressure fluctuation*

Kistler & Chen (1963), from their measurements of static pressure fluctuation measurements up to a Mach number $M_\infty$ of 5, have shown that the root-mean-square value of the pressure fluctuation is proportional to the local surface shear stress (see Kistler & Chen 1963, figure 16, p. 59). Thus

$$(\overline{p'^2})^{\frac{1}{2}}/\tau_w = \text{constant}.$$

The constant depends weakly on the free-stream Mach number $M_\infty$, and it changes from about 3 for subsonic boundary layers to about 5 for $M_\infty > 2$. This observation suggests that the pressure fluctuations are of the order of the local wall shear (square of friction velocity), i.e.

$$p' \sim U_*^2. \tag{12c}$$

### 3.4. *Density fluctuation*

For non-hypersonic boundary layers ($M_\infty < 5$) density fluctuation measurements have not been reported so far in the literature. However, for $M_\infty = 9$ the density fluctuations have been measured for the first time by Wallace (1969) using the 'electron beam technique'. In the absence of available experimental measurements for density fluctuations for the present case ($M_\infty < 5$), it may appear rather difficult to decide their order of magnitude. However, this is not so, as there are some indirect observations which can lead us to a decision about the order oı density fluctuations. Two such observations are the following.

(*a*) Laufer (1969) has pointed out that the temperature fluctuations are essentially isobaric, i.e.

$$t'/T = -\rho'/\rho,$$

a relation consistent with the observations of Kistler (1959) and Morkovin (1964). This relation implies (Laufer 1969) that the pressure fluctuations produced by vorticity-bearing velocity fluctuations are of higher order and can be neglected.

(*b*) Morkovin (1964) has shown that for the case when $(\gamma-1)M_\infty^2$ is of order unity ($M_\infty < 5$) the density fluctuation $\rho'/\rho$ is of the order of the velocity fluctuation $u'/U$, i.e.

$$\rho'/\rho \simeq (\gamma-1)\,M_\infty^2\,u'/U.$$

Observations (*a*) and (*b*) lead to the assumption that the fluctuations in $\rho'$, $u'$ and $T'$ are of equal order. In view of (12*a*) and (12*b*) it follows that

$$\rho' \sim U_*, \quad T_* \sim U_*. \qquad (12d, e)$$

The latter assumption (12*e*) can alternatively be explained as follows. In the definition of $T_*$, the heat flux $Q_w$ could be replaced by $\tau_w U_\infty$, which, as can be seen from Rotta's analysis, is roughly the maximum heat-transfer rate within the layer. Then

$$T_* = u_\tau U_\infty/C_p T_w \sim U_* \quad \text{if} \quad (\gamma-1)\,M^2 \sim 1.$$

Lastly, using the Reynolds or modified Reynolds analogy, it can be easily shown that in order for $T_*$ to be of same order as $U_*$ it is necessary for $T_w - T_r$ (where $T_r$ is the recovery temperature) to be of order $T_w$, and thus the assumption (12*e*) is valid for these large heat-transfer rates.

In the present work we shall employ the following notation for turbulent Reynolds terms. Second-order correlations of the type $-\overline{u'v'}$ will be denoted by $\tau_{uv}$ and third-order correlations of the type $-\overline{\rho'u'v'}$ by $\tau_{\rho uv}$. Thus we shall, in general, write

$$\tau_{ab} = -\overline{a'b'}, \quad \tau_{a_y b} = -\overline{a'_y b'} = -\overline{b'\,\partial a'/\partial y}, \quad \tau_{abc} = -\overline{a'b'c'}.$$

From our order assumption $a' \sim O(E_a)$ etc., it follows that

$$\tau_{ab} \sim O(E_a E_b), \quad \tau_{abc} \sim O(E_a E_b E_c). \qquad (13)$$

Here, the suffixes $a$, $b$ and $c$ are attached to $E$ for convenience of writing. The quantities, $E_u$, $E_v$, $E_\rho$, $E_t$ and $E_p$ are the scales of fluctuations in tangential velocity, normal velocity, density, static temperature and pressure and without loss of generality can be written as

$$\left.\begin{aligned} &E_u = E_v = E_\rho = U_{*0} \quad (\equiv E \text{ say}),\\ &E_t = T_{*0}, \quad E_t = A_t E,\\ &E_p = U_{*0}^2, \end{aligned}\right\} \qquad (14)$$

where $U_{*0}$ and $T_{*0}$ are the friction velocity and temperature at some designated point $x = x_0$. The above relations (14) are the direct consequence of observations (12*a*–*d*) that the fluctuations in velocities, temperature and density are of the order of the friction velocity and fluctuations in pressure are of the order of the local shear stress. Lastly the fluctuations in viscosity are assumed to be of the order of fluctuations in temperature $t'$, i.e. $T_{*0}$.

From the behaviour of the fluctuations (equations (14)) and the arguments that follow, it is obvious that the appropriate gauge function for an asymptotic expansion for the mean velocity field is the friction velocity. Furthermore, the second-order effects in turbulent boundary layers when compared with first-order effects are again of the order of friction velocity, as can be seen from the following.

(i) Measurements of Lobb, Winkler & Persh quoted by Schlichting (1968, figure 23.11) show that the characteristic shape of the law of the wall $\bar{u}/u_\tau$ *vs.* $\bar{y}u_\tau/\bar{\nu}_w$ is similar to that of an incompressible flow except that it weakly depends upon the friction Mach number $M_\tau$.

(ii) The analysis of Rotta (1960) also shows that the law of the wall for a compressible flow depends on the friction Mach number $M_\tau$. The effect of $M_\tau$ on the law of the wall is weak (a fact pointed out by Rotta 1967). The maximum value of $M_\tau$ encountered by Rotta (1960) is 0·12 when $M_\infty = 5$ and 0·14 when $M_\infty = 8$.

Now in view of (i) and (ii) it follows that the law of the wall depends upon $M_\tau$, which goes to zero like $U_*$ as $R \to \infty$. Thus it is preferable to regard the weak effects of $M_\tau$ (of the order of $U_*$) on the law of the wall as second-order effects. Thus the appropriate sequence of gauge functions is

$$1, E, E^2, \ldots. \tag{15}$$

## 4. Analysis at large Reynolds number

We shall now analyse the underdetermined set of equations of mean motion (4)–(11) for turbulent boundary-layer flow of a compressible fluid at large Reynolds number. The parameters $\sigma$ and $D$ are assumed to be of order unity. The method used is that of matched asymptotic expansions and needs three limiting processes, an inviscid limit (defined by $y$ fixed, $R \to \infty$), an outer limit $[Y = y/\Delta$ fixed, $R \to \infty$; $\Delta \sim O(E)]$ and an inner limit $[\eta = y/\delta$ fixed, $R \to \infty$; $\delta \sim O(E^{-1}R^{-1})]$, for describing the flow. In this section, we shall study (4)–(11) in each of the three limits and try to match in their regions of common validity.

Before we proceed, let us examine the equation of state (10*b*) for fluctuations. It will be shown later that throughout the boundary layer $T$ and $\rho$ are of order unity, and under our assumptions $\rho'$, $t' \sim O(E)$ and $p' \sim O(E^2)$, equation (10*b*) gives to the lowest order

$$\rho'/\rho + t'/T = 0. \tag{16}$$

This is the familiar equation of state for fluctuations proposed by various authors empirically (see Kutateladze & Leont'ev 1964; Laufer 1969).

### 4.1. *Inviscid layer*

The inviscid limit is defined as $y$, $\sigma$ and $D$ fixed as $R \to \infty$. As was described earlier, the turbulent boundary layer has three length scales: an inviscid scale, an outer scale $\Delta \sim O(E)$ and an inner scale $\delta \sim O(E^{-1}R^{-1})$. The ratio of the outer to the inviscid length scale is $\Delta \sim O(E)$ and of the inner to the outer scale is

$$\hat{\delta} = \delta/\Delta \sim O(E^2R)^{-1}.$$

Thus the structure of the turbulent boundary layer depends upon two small parameters $E$ and $\hat{\delta}$ and the appropriate inviscid expansion (Van Dyke 1964) for any of the variables, say $U$, is of the form

$$U(x, y; R) = U_1(x, y; \hat{\delta}) + EU_2(x, y; \hat{\delta}) + E^2U_3(x, y; \hat{\delta}) + \ldots. \tag{17a}$$

As will be shown later $E \sim O(\ln \hat{\delta})^{-1}$ as $\hat{\delta} \to 0$, i.e. $\hat{\delta}$ is of much higher order than $E$. Therefore the terms $U_1(x,y;\hat{\delta}), U_2(x,y;\hat{\delta}), \ldots$, may be represented by $U_1(x,y,0)$, $U_2(x,y,0), \ldots$, in the above expansion. Therefore the expansion can be written as

$$U = U_1(x,y) + EU_2(x,y) + E^2U_3(x,y) + \ldots \\ + \hat{\delta}[\tilde{U}_1(x,y) + E\tilde{U}_2(x,y) + \ldots] + \ldots. \quad (17b)$$

Thus we consider the following inviscid expansions:

$$\left.\begin{aligned} U &= U_1(x,y) + EU_2(x,y) + E^2U_3(x,y) + O(E^3), \\ V &= V_1(x,y) + EV_2(x,y) + E^2V_3(x,y) + O(E^3), \\ P &= P_1(x,y) + EP_2(x,y) + E^2P_3(x,y) + O(E^3), \\ \rho &= \rho_1(x,y) + E\rho_2(x,y) + E^2\rho_3(x,y) + O(E^3), \\ T &= T_1(x,y) + E_tT_2(x,y) + E_t^2T_3(x,y) + O(E^3), \\ \tau_{ab} &= E_aE_b\Gamma_{ab_1}(x,y) + O(E_a^2E_b), \\ \tau_{abc} &= E_aE_bE_c\Gamma_{abc_1}(x,y) + O(E_a^2E_bE_c). \end{aligned}\right\} \quad (18)$$

By substituting the expansions (18) into the equations of mean motion (4)–(10) and collecting the coefficient of like powers of $E$ we get the problem for successive orders. The equations to lowest order are

$$\left.\begin{aligned} (\rho_1U_1)_x + (\rho_1V_1)_y = 0, \qquad \rho_1Z_1(U_1) + P_{1x} = 0, \\ \rho_1Z_1(V_1) + P_{1y} = 0, \quad \rho_1Z_1(T_1) - DZ_1(P_1) = 0, \\ P_1 = [(\gamma-1)/\gamma]\rho_1T_1/D. \end{aligned}\right\} \quad (19)$$

Here $Z_m$ is an operator given by

$$Z_m = U_m\partial/\partial x + V_m\partial/\partial y. \quad (20)$$

Equations (19) are the well-known Euler equations of motion. Integration of these equations along a streamline gives

$$T_1 + \tfrac{1}{2}D(U_1^2 + V_1^2) = G_1(\Psi_1), \quad (21a)$$

$$\rho_1 = [(\gamma-1)T_1/\gamma]^{1/(\gamma-1)}\exp[\gamma S_1(\Psi_1)/(\gamma-1)], \quad (21b)$$

$$P_1 = \rho_1^\gamma\exp[\gamma S_1(\Psi_1)]. \quad (21c)$$

These expressions show that the entropy $S_1$ and the total enthalpy $G_1$ are constant along a streamline. For a uniform oncoming stream, it follows that $S_1$ and $G_1$ are constant throughout the outer region. The first-order equations are

$$\left.\begin{aligned} (\rho_1U_2 + \rho_2U_1)_x + (\rho_1V_2 + \rho_2V_1)_y = 0, \\ \rho_1Z_2(U_1) + \rho_1Z_1(U_2) + \rho_2Z_1(U_1) + P_{2x} = 0, \\ \rho_1Z_2(V_1) + \rho_1Z_1(V_2) + \rho_2Z_1(V_1) + P_{2y} = 0, \\ \rho_1Z_2(T_1) + \rho_1A_tZ_1(T_2) + \rho_2Z_1(T_1) - D[Z_2(P_1) + Z_1(P_2)] = 0, \\ P_2 = [(\gamma-1)/\gamma](\rho_2T_1 + \rho_1T_2A_t)/D. \end{aligned}\right\} \quad (22)$$

For the second approximation one obtains a corresponding, more complicated, system of equations, of which those for continuity and tangential momentum are

$$(\rho_1 U_3+\rho_2 U_2+\rho_3 U_1-\Gamma_{\rho u_1})_x+(\rho_1 V_3+\rho_2 V_2+\rho_3 V_1-\Gamma_{\rho v_1})_y = 0, \qquad (23a)$$

$$\rho_1 Z_1(U_3)+\rho_1 Z_3(U_1)+\rho_1 Z_2(U_2)+\rho_2 Z_2(U_1)+\rho_2 Z_1(U_2)+\rho_3 L_1(U_1)+P_{3x}$$
$$=(\rho_1 \Gamma_{uu_1}+U_1\Gamma_{\rho u_1})_x+(\rho_1\Gamma_{uv_1}+V_1\Gamma_{\rho u_1})_y. \qquad (23b)$$

Equations (23) for the second approximation involve all the second-order correlations in velocities, density and temperature.

Equations (19), (22) and (23) are 'inviscid' in the sense that they do not involve viscous and heat-conduction terms. Thus, the inviscid equations, in general, cannot satisfy the no-slip condition.

### 4.2. *Outer layer*

The outer limit is defined as $Y = y/\Delta$, $\sigma$ and $D$ fixed as $R \to \infty$. The behaviour of the inviscid expansions (18) for small $y$ leads us to study the following outer expansions:

$$\left.\begin{aligned}
U(x,y;R) &= u_1(x, Y)+Eu_2(x, Y)+E^2u_3(x, Y)+O(E^3),\\
V &= v_1(x, Y)+Ev_2(x, Y)+E^2v_3(x, Y)+O(E^3),\\
P &= P_1(x, Y)+Ep_2(x, Y)+E^2p_3(x, Y)+O(E^3),\\
\rho &= \tilde{\rho}_1(x, Y)+E\tilde{\rho}_2(x, Y)+E^2\tilde{\rho}_3(x, Y)+O(E^3),\\
T &= h_1(x, Y)+E_t h_2(x, Y)+E_t^2 h_3(x, Y)+O(E^3),\\
\tau_{ab} &= E_a E_b \tau_{ab_1}(x, Y)+E_a E_b E\tau_{ab_2}(x, Y)+O(E^4),\\
\tau_{abc} &= E_a E_b E_c \tau_{abc_1}(x, Y)+O(E^4).
\end{aligned}\right\} \qquad (24)$$

On substituting the expansions (24) in (4)–(10) and equating to zero the coefficient of lowest power of $E$, we get the lowest order problem

$$(\tilde{\rho}_1 u_1)_x+(\tilde{\rho}_1 v_1)_y = 0, \qquad (25a)$$

$$\tilde{\rho}_1(u_1 u_{1x}+v_1 u_{1Y})+P_{1x} = 0, \qquad (25b)$$

$$p_{1Y} = 0, \qquad (25c)$$

$$\tilde{\rho}_1(u_1 h_{1x}+v_1 h_{1Y})-Du_1 p_{1x} = 0, \qquad (25d)$$

$$p_1 = [(\gamma-1)/\gamma]\,\tilde{\rho}_1 h_1/D. \qquad (25e)$$

The problem to next order shows that $\Delta$ is of the order of $E$. Without loss of generality let $\Delta = E$. Now the first-order problem is governed by following equations:

$$(\tilde{\rho}_1 u_2+\tilde{\rho}_2 u_1)_x+(\tilde{\rho}_1 v_2+\tilde{\rho}_2 v_1-\tau_{\rho v_1})_Y = 0, \qquad (26a)$$

$$(\tilde{\rho}_1 u_2+\tilde{\rho}_2 u_1)u_{1x}+(\tilde{\rho}_1 v_2+\tilde{\rho}_2 v_1-\tau_{\rho v_1})\,u_{1Y}+\tilde{\rho}_1(u_1 u_{2x}+v_1 u_{2Y})+P_{2x}$$
$$=(\tilde{\rho}_1\tau_{uv_1})_Y, \qquad (26b)$$

$$P_{2Y} = 0, \qquad (26c)$$

$$(\tilde{\rho}_1 u_2+\tilde{\rho}_2 u_1)\,h_{1x}+(\tilde{\rho}_1 v_2+\tilde{\rho}_2 v_1-\tau_{\rho v_1})\,h_{1Y}+\tilde{\rho}_1 A_t(u_1 h_{2x}+v_1 h_{2Y})$$
$$-D(u_2 P_{1x}+u_1 p_{2x}) = (\tilde{\rho}_1\tau_{vt_1})_Y, \qquad (26d)$$

$$p_2 = [(\gamma-1)/\gamma]\,(\tilde{\rho}_1 h_2 A_t+\tilde{\rho}_2 h_1)/D. \qquad (26e)$$

The equation of state (11*b*) for the fluctuations gives

$$\tau_{\rho v_1} = -\tilde{\rho}_1 A_t \tau_{vt_1}/h_1. \tag{26f}$$

For the second approximation, we obtain a corresponding more complicated system of equations:

$$(\tilde{\rho}_1 u_3 + \tilde{\rho}_2 u_2 + \tilde{\rho}_3 u_1 - \tau_{\rho u_1})_x + (\tilde{\rho}_1 v_3 + \tilde{\rho}_2 v_2 + \tilde{\rho}_3 v_1 - \tau_{\rho v_2})_Y = 0, \tag{27a}$$

$$\begin{aligned}(\tilde{\rho}_1 u_3 + \tilde{\rho}_2 u_2 + \tilde{\rho}_3 u_1 - \tau_{\rho u_1}) u_{1x} + (\tilde{\rho}_1 v_3 + \tilde{\rho}_2 v_2 + \tilde{\rho}_3 v_1 - \tau_{\rho v_2}) u_{1Y} \\ + (\tilde{\rho}_1 u_2 + \tilde{\rho}_2 u_1) u_{2x} + (\tilde{\rho}_1 v_2 + \tilde{\rho}_2 v_1) u_{2Y} + \tilde{\rho}_1 (u_1 u_{3x} + v_2 u_{3Y}) + P_{2x} \\ = [\tilde{\rho}_1 \tau_{uv_2} + \tilde{\rho}_2 \tau_{uv_1} + v_1 \tau_{\rho u_1} + \tau_{\rho u v_1}]_Y + (\tilde{\rho}_1 \tau_{uu_1} + u_1 \tau_{\rho u_1})_x,\end{aligned} \tag{27b}$$

$$P_{3Y} = (\tilde{\rho}_1 \tau_{vv_1})_Y. \tag{27c}$$

In the similar way we can write equations for energy and state. These outer equations (25*c*), (26*c*) and (27*c*) show that the pressure is constant across the outer layer to the order $E$. Further, the first-order outer equations involve double correlations $\tau_{uv_1}$ and $\tau_{vt_1}$, while the second-order equations (27) involve third-order correlations $\tau_{\rho uv_1}$ and $\tau_{\rho vt_1}$, along with other second-order correlations. Up to this order the outer-flow equations are independent of viscous dissipation and turbulent work of compressibility due to correlations between pressure and velocity fluctuations $\overline{vp'_y}$.

These outer equations do not involve viscous and heat-conduction terms and thus the outer expansion will also fail near the wall in that it will not satisfy the no-slip condition.

### 4.3. *Matching of expansions for inviscid and outer layers*

The inviscid expansion (17) and the outer expansions (24) will now be matched in the region where both are valid (this overlap region is not the same as Millikan's overlap), by the use of the well-known matching principle (see Van Dyke 1964)

$$\mathcal{N}_m \mathcal{O}_n(f) = \mathcal{O}_n \mathcal{N}_m(f), \tag{28}$$

where $\mathcal{N}_m(f)$ and $\mathcal{O}_m(f)$ represent the $m$-term inviscid and outer expansions of $f$. The left-hand side of the above matching principle can be obtained by first writing the $n$-term outer expansion of $f$ in terms of the inviscid variable $y$ and then using the $m$-term inviscid expansion. Similarly, the right-hand side of the relation (28) is obtained by first taking $m$ terms of the inviscid expansion of $f$ written in terms of outer variable $Y$ and then using the $n$-term outer expansion.

If the matching principle with $m = n = 1$ is applied, the expansions (17) and (24) give

$$\left.\begin{aligned} u_1(x,Y) &= U_1(x,0), & p_1(x,Y) &= P_1(x,0), \\ \rho_1(x,Y) &= \tilde{\rho}_1(x,0), & h_1(x,Y) &= T_1(x,0), \\ \tau_{ab_1}(x,Y) &= \Gamma_{ab_1}(x,0), & \tau_{abc_1}(x,Y) &= \Gamma_{abc_1}(x,0), \end{aligned}\right\} \quad \text{as} \quad Y \to \infty. \tag{29}$$

For later convenience the quantities $U_1(x,0)$, $P_1(x,0)$, $\rho_1(x,0)$ and $T_1(x,0)$ will be denoted by $U_{10}$, $P_{10}$, $\rho_{10}$ and $T_{10}$ respectively. Note that no condition has been imposed on the $V$ component of velocity. Next, matching with $n = 1$ and $m = 2$ gives the matching condition

$$V_1(x,0) = 0, \tag{30a}$$
$$V_2(x,0) = \lim_{Y\to\infty} [v_1 - Y v_{1Y}]. \tag{30b}$$

Equation (30*b*) is the matching condition for the first-order inviscid flow and shows that the effects of displacement of the inviscid solution by the outer solution is that of the surface distribution of sources.

Now the solution of the lowest order outer equation (25) which satisfies the matching conditions (29) is

$$\left.\begin{aligned} u_1 &= U_{10}, \quad \tilde{\rho}_1 = \rho_{10}, \\ v_1 &= [Q - (\rho_{10}U_{10})_x\, Y]/\rho_{10}, \\ p_1 &= P_{10}, \quad h_1 = T_{10}. \end{aligned}\right\} \tag{31}$$

Here $Q$ is the constant of integration and will be determined, later, by matching (31) with the inner solution. Furthermore, with the help of (31) the result (30*b*) becomes

$$V_2(x,0) = Q/\rho_{10}. \tag{32}$$

We now match first-order terms in the expansions (18) and (24), and using the matching principle with $m = n = 2$ we get

$$\left.\begin{aligned} u_2(x,Y) &= U_2(x,0) + Y\, U_{1y}(x,0), \\ v_2(x,0) &= V_2(x,0) + Y\, V_{1y}(x,0), \\ &\;\vdots \qquad\qquad \vdots \\ \tau_{ab}(x,Y) &= \Gamma_{ab_2}(x,0) + Y\, \Gamma_{ab_1y}(x,0), \end{aligned}\right\} \quad \text{as} \quad Y \to \infty. \tag{33}$$

The derivatives $U_{1y}$ etc. may be evaluated from the inviscid equations (19)–(21). For the present case of a non-uniform free stream, we have at $y = 0$

$$\left.\begin{aligned} U_{1y} &= -\rho_{10}[T_{10}S_1'(0) - G_1'(0)], \\ V_{1y} &= -(\rho_{10}U_{10})_x/\rho_{10}, \\ P_{1y} &= 0, \\ \rho_{1y} &= -\rho_{10}^2 U_{10} S_1'(0), \\ T_{1y} &= -\rho_{10}U_{10}T_{10}S_1'(0). \end{aligned}\right\} \tag{34}$$

In writing the relation for $U_{1y}$, we have used Crocco's vortex theorem. Here $S_1' = dS_1/d\Psi_1$, $G_1' = dG_1/d\Psi_1$ and $\Psi_1$ is the first-order inviscid stream function defined by continuity equation in (19).

Let us now consider the second-order flow. Applying the matching principle with $n = 2$ and $m = 3$ gives the matching condition

$$V_3(x,0) = v_2 - Y v_{2Y} - \tfrac{1}{2} Y^2 v_{2YY} \quad \text{as} \quad Y \to \infty, \tag{35}$$

which represents the effects of displacement of the first-order outer flow by the second-order inviscid solution. Lastly, the matching principle with $m = 3$ and

$n = 3$ gives the matching conditions for the second-order outer flow. The condition for the tangential velocity component is

$$u_3(x, Y) = U_3(x, 0) + YU_{2y}(x, 0) + \tfrac{1}{2}Y^2 U_{1yy}(x, 0) \quad \text{as} \quad Y \to \infty. \tag{36}$$

Conditions similar to (36) can be written down for other variables.

### 4.4. *Inner layer*

Failure of the outer limit near the wall requires the introduction of an inner limit. From an order-of-magnitude analysis we are led to introduce the inner variable

$$\eta = yER/\nu_w$$

(where $\nu_w$ is the non-dimensional local kinematic viscosity at the wall) and study the limit $R \to \infty$ for fixed $\eta$, $\sigma$ and $D$. The inner limit of the outer expansions (24) shows that the appropriate inner expansions are

$$\left.\begin{aligned} U &= E\hat{u}_2(x,\eta) + E^2\hat{u}_3(x,\eta) + O(E^3),\\ V &= E\hat{v}_2(x,\eta) + E^2\hat{v}_3(x,\eta) + O(E^3),\\ P &= \hat{p}_1(x,\eta) + E\hat{p}_2(x,\eta) + E^2\hat{p}_3(x,\eta) + O(E^3),\\ \rho &= \hat{\rho}_1(x,\eta) + E\hat{\rho}_2(x,\eta) + E^2\hat{\rho}_3(x,\eta) + O(E^3),\\ T &= \hat{h}_1(x,\eta) + E_t\hat{h}_2(x,\eta) + E_t^2\hat{h}_3(x,\eta) + O(E^3),\\ \tau_{ab} &= E_a E_b \hat{\tau}_{ab_1}(x,\eta) + E_a E_b E \hat{\tau}_{ab_2}(x,\eta) + O(E_a E_b E^2),\\ \tau_{abc} &= E_a E_b E_c \hat{\tau}_{abc_1}(x,\eta) + O(E_a^2 E_b E_c). \end{aligned}\right\} \tag{37}$$

Furthermore, the viscosity $\mu$ is function of temperature and also requires an appropriate inner expansion. The Taylor series expansion of $\mu$ around $\hat{h}_1$ gives

$$\mu = \mu(\hat{h}_1) + E_t \mu_T(\hat{h}_1)\,\hat{h}_2 + O(E_t^2),$$

where $\mu_T = (d\mu/dT)_{T=\hat{h}_1}$.

Introduction of these inner expansions (37) into the equations of mean motion (4)–(10) and collection of coefficients of various powers of $E$ give the equations for successive approximations. To the lowest order, the equations of continuity, normal momentum and state give

$$(\hat{\rho}_1\hat{v}_1)_\eta = 0, \quad \hat{p}_{1\eta} = 0, \quad \hat{p}_1 = [(\gamma-1)/\gamma]\,\hat{\rho}_1\hat{h}_1/D. \tag{38$a$–$c$}$$

Equation (38$b$) shows that the pressure is constant across the inner region. Now, matching the pressure of the inner region (28$b$) with that of the outer region (18) to lowest order, we get

$$\hat{\rho}_1\hat{h}_1 = \tilde{\rho}_1 h_1 = \rho_{10} T_{10} = \rho_w T_w. \tag{39}$$

Integration of (38$a$) gives

$$\hat{\rho}_1\hat{v}_1 = \rho_w v_w, \tag{40}$$

where $v_w$ is the velocity of transpiration at the wall, if any. The matching of the normal component of velocity in the inner region (40) with that in the outer region (31$c$) determines $Q = \rho_w v_w$, and (32) reduces to

$$V_2(x, 0) = v_w(x)\,T_{10}/T_w. \tag{41}$$

Equation (41) is the essential boundary condition for the solution of the displacement-speed problem (22). This shows that, in presence of transpiration ($v_w \neq 0$), the problem is global in nature, in the sense that the displacement-speed problem (22) has a non-trivial solution. However, for an impermeable wall ($v_w = 0$) the boundary condition (41) becomes homogeneous, and the solution of the displacement-speed problem (22) is trivial and hence the effects of displacement are of higher order. For the latter case, by matching the pressure in the inner, outer and inviscid regions to order $E$ we get

$$\hat{p}_2(x,\eta) = p_2(x, Y) = P_2(x,y) = 0. \tag{42}$$

Now for the case of no transpiration the $x$ momentum equation, energy equation and equation of state give to the lowest order $\hat{\rho}_1 = \rho_w$ and $\hat{h}_1 = T_w$ and to first order

$$(\hat{u}_{2\eta} + \hat{\tau}_{uv_1})_\eta = 0, \tag{43a}$$

$$(\sigma^{-1}\hat{h}_{2\eta} + \hat{\tau}_{vt_1})_\eta = 0, \tag{43b}$$

$$\hat{\rho}_2 = -(\rho_w/T_w)\,A_t\hat{h}_2. \tag{43c}$$

The equation of state (11*b*) for fluctuations leads to

$$\hat{\tau}_{\rho v_1} = -\rho_w A_t \hat{\tau}_{vt_1}/T_w. \tag{43d}$$

For the second approximation, we obtain a corresponding, more complicated, system of equations:

$$\rho_w \hat{v}_{2\eta} = \hat{\tau}_{\rho v_1 \eta}, \tag{44a}$$

$$[\hat{u}_{3\eta} + \hat{\tau}_{uv_2} + (\hat{\rho}_2\hat{\tau}_{uv_1} + \hat{\tau}_{\rho uv_1})/\rho_w + \hat{h}_2 A_t \hat{u}_{2\eta}\mu_T/\mu - A_t\mu^{-1}\hat{\tau}_{\mu u_\eta 1}]_\eta - \hat{v}_2\hat{u}_{2\eta} = 0, \tag{44b}$$

$$\hat{p}_{3\eta} = \rho_w \hat{\tau}_{vv_1\eta}, \tag{44c}$$

$$[\sigma^{-1}A_t\hat{h}_{3\eta} + \hat{\tau}_{vt_2} + (\hat{\rho}_2\hat{\tau}_{vt_1} + \hat{\tau}_{\rho vt_1})/\rho_w + \sigma^{-1}A_t\hat{h}_2\hat{h}_{2\eta}\mu_T/\mu - (\mu\sigma)^{-1}\hat{\tau}_{\mu t_\eta 1}]_\eta$$
$$- \hat{v}_2\hat{h}_{2\eta} = D[\hat{\tau}_{p_\eta v_1} - \rho_w(\hat{u}_{2\eta}^2 + \hat{\tau}_{u_\eta u_\eta 1})], \tag{44d}$$

$$p_3 = \{(\gamma-1)/\gamma\}\{\hat{\rho}_3 T_w + A_t\hat{\rho}_2\hat{h}_2 + A_t^2\rho_w\hat{h}_3 - A_t\hat{\tau}_{\rho t_1}\}/D. \tag{44e}$$

The equations of first order, equations (43), are independent of convection and viscous dissipation and involve only second-order correlations $\hat{\tau}_{uv_1}$ and $\hat{\tau}_{vt_1}$. It may be noted that the equations of second order, equations (44), contain the convection terms due to the normal velocity $v_2$, caused by turbulent mass flux $\hat{\tau}_{\rho v_1}$. For an incompressible flow, however, this turbulent mass flux is zero and convection is a higher order effect. Furthermore, (44) involves viscous dissipation, triple correlations and the work of compressibility $\hat{\tau}_{p_\eta v_1}$ due to correlation between velocity and pressure fluctuations. Lastly, these equations for the inner region show that the pressure is constant across the inner layer.

The solution to (44*a*) which satisfies the boundary condition at a solid wall is

$$\hat{v}_2 = \hat{\tau}_{\rho v_1}/\rho_w. \tag{45}$$

Now, using (43*c*), (43*d*) and (45) and integrating the tangential momentum equation (44*b*) and temperature equation (44*d*) once we get

$$\hat{u}_{3\eta} + \hat{\tau}_{uv_2} + \frac{\hat{h}_2 A_t}{T_w}\left(\frac{T_w\mu_T}{\mu}\hat{u}_{2\eta} - \hat{\tau}_{uv_1}\right) + \frac{1}{\rho_w}\hat{\tau}_{\rho uv_1} - A_t\mu^{-1}\hat{\tau}_{\mu u_\eta 1}$$
$$+ \frac{A_t}{T_w}\int_0^\eta \hat{\tau}_{vt_1}\hat{u}_{2\eta}\,d\eta = f_2, \tag{46}$$

$$\sigma^{-1}A_t\hat{h}_{3\eta}+\hat{\tau}_{vt_2}+\frac{\hat{h}_2 A_t}{T_w}\left(\frac{T_w\mu_T}{\mu}\hat{h}_{2\eta}-\hat{\tau}_{vt_1}\right)+\frac{1}{\rho_w}\tau_{\rho vt_1}-(\mu\sigma)^{-1}\hat{\tau}_{\mu t_\eta 1}$$
$$+\frac{A_t}{T_w}\int_0^\eta \hat{\tau}_{vt_1}\hat{h}_{2\eta}\,d\eta = D\int_0^\eta[\hat{\tau}_{p_\eta v_1}-\rho_w(\hat{u}_{2\eta}^2+\hat{\tau}_{u_\eta u_\eta 1})]\,d\eta+g_2. \quad (47)$$

Here $f_2$ and $g_2$ are constants of integration and represent second-order contributions respectively to skin friction and heat transfer at the wall.

The equations to first and second order show that the law of the wall, for example, for velocity profile in functional and form can be written as

$$u = Eu_1(\eta)+E^2u_2(\eta, A_t, M_\infty, \sigma, \gamma, \omega)+O(E^2). \quad (48)$$

This may be compared with the relation (3) obtained from the work of Rotta. The mean tangential momentum equation shows that effects of the pressure gradient are governed by the parameter $\pi_p = E^{-3}R^{-1}P_{1x}$. For a given pressure distribution and a given location other than a point of zero skin friction, this term approaches zero as the Reynolds number approaches infinity. In a separating boundary layer, the pressure term does not approach zero uniformly. As a result the expansions used will not be valid near a separation point.

Finally, the dissipation in the mean temperature equations is governed by the parameter $\pi_M = (\gamma-1)M_\infty^2 E$. In the present work, where $(\gamma-1)M_\infty^2$ is assumed to be of order unity, the parameter $\pi_M$ goes to zero uniformly as $R\to\infty$. Thus the effects of viscous dissipation are of second order. It also turns out that the work of compressibility due to fluctuations $\overline{v'p'_y}$ is also of second order, showing that the pressure fluctuations do not influence the boundary layer to lowest order.

### 4.5. *Matching of outer and inner expansions*

The matching of the pressure and the normal component of velocity has been carried out in §4.4. We now match the expansions for tangential velocity, temperature and density in the outer and inner expansions, (24) and (37) respectively, in the overlap region (Millikan 1939) by the matching condition (Van Dyke 1964)

$$\mathscr{I}_m\mathscr{O}_n(f) = \mathscr{O}_n\mathscr{I}_m(f). \quad (49)$$

We first consider the matching of turbulent Reynolds terms. Applying the matching principle with $m = 1$ and $n = 1$, the outer expansions (24) and inner expansion (37) to double and triple correlations give

$$\hat{\tau}_{ab_1}(x,\eta) \sim \tau_{ab_1}(x,0) \quad \text{as} \quad \eta\to\infty,$$
$$\hat{\tau}_{abc_1}(x,\eta) \sim \tau_{abc_1}(x,0) \quad \text{as} \quad \eta\to\infty.$$

Next, applying the matching principle with $m = n = 2$, the second-order correlations give

$$\hat{\tau}_{ab_2}(x,\eta) \sim \tau_{ab_2}(x,0) \quad \text{as} \quad \eta\to\infty.$$

Therefore, without loss of generality, we assume that

$$\hat{\tau}_{ab_1} \sim I_{ab_1}(x)-J_{ab_1}(x)/\eta+\ldots \quad \text{as} \quad \eta\to\infty, \quad (50a)$$
$$\hat{\tau}_{ab_2} \sim I_{ab_2}(x)-J_{ab_2}(x)/\eta+\ldots \quad \text{as} \quad \eta\to\infty, \quad (50b)$$
$$\hat{\tau}_{abc_1} \sim I_{abc_1}(x)-J_{abc_1}(x)/\eta+\ldots \quad \text{as} \quad \eta\to\infty. \quad (50c)$$

Integration of the first-order inner equations (43) and use of (50) and boundary conditions at the wall give

$$u_2 = (f_1 - I_{uv_1})\,\eta + J_{uv_1}\ln(1+\eta) + H_{uv_1}(\eta), \tag{51a}$$

$$\sigma^{-1}h_2 = (g_1 - I_{vt_1})\,\eta + J_{vt_1}\ln(1+\eta) + H_{vt_1}(\eta), \tag{51b}$$

where $f_1$ and $g_1$ are unspecified functions of integration and represent respectively the first-order skin friction and heat transfer at the wall. The function $H_{ab_1}$ is a regular integral defined by

$$H_{ab_1} = \int_0^\eta I_{ab_1} - \frac{J_{ab_1}}{1+\eta} - \hat{\tau}_{ab_1}\,d\eta. \tag{52}$$

For the matching of tangential velocity component and temperature, we first consider $\mathscr{I}_2(U)$ and $\mathscr{I}_2(T)$ as given by (51a) and (51b):

$$\mathscr{I}_2(U) \sim E[(f_1 - I_{uv_1})\,YE^2R/\nu_w + J_{uv_1}\ln(YE^2R/\nu_w) + H_{uv_1}] \quad \text{as} \quad \eta \to \infty, \tag{53a}$$

$$\mathscr{I}_2(T) \sim T_w + E_t[(g_1 - I_{vt_1})\,Y\sigma E^2R/\nu_w + J_{vt_1}\sigma\ln(YE^2R/\nu_w) + \sigma H_{vt_1}] \quad \text{as} \quad \eta \to \infty. \tag{53b}$$

Applying the matching principle (47) with $m = n = 2$ shows that

$$f_1 = I_{uv_1}, \quad g_1 = I_{vt_1}, \tag{54a, b}$$

$$E\ln(E^2R/\nu_w) \sim O(1), \quad E_t\ln(E^2R/\nu_w) \sim O(1). \tag{55a, b}$$

$\mathscr{I}_3(U)$ is given by (46), (50) and (53a) as

$$\begin{aligned}\mathscr{I}_3(U) \sim{}& E[J_{uv_1}\ln(YE^2R/\nu_w) + H_{uv_1}] + E^2[(\chi_1 - \chi_3)\,YE^2R/\nu_w \\ &+ \chi_3(E^2RY/\nu_w)\ln(YE^2R/\nu_w) + \chi_2\ln(YE^2R/\nu_w) + \chi_4\{\ln(YE^2R/\nu_w)\}^2 + \Lambda_2],\end{aligned} \tag{56}$$

where

$$\left.\begin{aligned}\chi_1 &= f_2 - I_{uv_2} - \rho_w^{-1}I_{\rho uv_1} + \mu^{-1}I_{\mu u_\eta 1}A_t + (I_{uv_1}H_{vt_1} + J_{uv_1}J_{vt_1})\,A_t/T_w + \Lambda_1,\\ \chi_2 &= J_{uv_2} + \rho_w^{-1}J_{\rho uv_1} - \mu^{-1}J_{\mu u_\eta 1}A_t - A_tJ_{uv_1}J_{vt_1}/T_w\\ &\quad - A_t(1+\mu_T T_w/\mu)\,J_{uv_1}H_{vt_1}/T_w,\\ \chi_3 &= (J_{vt_1}I_{uv_1} - J_{uv_1}I_{vt_1})\,A_t/T_w,\\ \chi_4 &= -(1+\mu_T T_w/\mu)\,J_{uv_1}J_{vt_1}A_t/2T_w,\end{aligned}\right\} \tag{57}$$

and $\Lambda_1$ and $\Lambda_2$ are the regular and bounded integrals for $\eta \to \infty$. The matching condition applied with $m = n = 3$ gives

$$\chi_1 = 0, \quad \chi_3 = 0, \tag{58a, b}$$

and $E\ln(E^2R/\nu_w)$ has to be of order unity from (55a). In order to obtain the results in a convenient form, let

$$[1 + E\chi_2/J_{uv_1} + \ldots]\,E\ln(E^2R/\nu_w) = \alpha_1 + \alpha_2E + \alpha_3E^2 + O(E^2). \tag{59}$$

Introducing (59) in (56) we get

$$\begin{aligned}\mathscr{O}_3\mathscr{I}_3(U) \sim{}& \alpha_1J_{uv_1} + \chi_4\alpha_1^2 + E[(J_{uv_1} + 2\chi_4\alpha_1)\ln Y + H_{uv_1} + \alpha_2J_{uv_1}\\ &+ 2\chi_4\alpha_1(\alpha_2 - \alpha_1\chi_2/J_{uv_1})] + E^2[\chi_4(\ln Y)^2 + (\chi_2 + 2\chi_4\alpha_2\\ &- 2\alpha_1\chi_4\chi_2/J_{uv_1})\ln Y + \alpha_3J_{uv_1} + \Lambda_2 + \chi_4\Lambda_3].\end{aligned} \tag{60}$$

Applying the matching principle (47) with $m = n = 3$ we get

$$\alpha_1 J_{uv_1} + \chi_4 \alpha_1^2 = u_1(x, 0), \tag{61a}$$

$$u_2(x, Y) \sim (J_{uv_1} + 2\chi_4\alpha_1)\ln Y + H_{uv_1} + 2\chi_4\alpha_1(\alpha_2 - \alpha_1\chi_2/J_{uv_1}) + \alpha_2 J_{uv_1} \quad \text{as} \quad Y \to 0, \tag{61b}$$

$$u_3(x, Y) \sim \chi_4(\ln Y)^2 + (\chi_2 + 2\chi_4\alpha_2 - 2\alpha_1\chi_4\chi_2/J_{uv_1})\ln Y + \alpha_3 J_{uv_1} + \Lambda_2 + \chi_4\Lambda_3] \quad \text{as} \quad Y \to 0. \tag{61c}$$

Similarly, for the temperature profile equations (47) gives

$$\begin{aligned}\mathscr{I}_3(T) \sim T_w + E_t[\sigma J_{vt_1}\ln(YE^2R/\nu_w) + H_{vt_1}] + E_t^2[\hat{\chi}_0 Y^2E^4R^2/\nu_w^2 \\ + (\hat{\chi}_1 - \hat{\chi}_3)YE^2R/\nu_w + \hat{\chi}_3(YE^2R/\nu_w)\ln(YE^2R/\nu_w) \\ + \hat{\chi}_2\ln(YE^2R/\nu_w) + \hat{\chi}_4\{\ln(YE^2R/\nu_w)\}^2 + \Lambda_{2t}],\end{aligned} \tag{62}$$

where

$$\hat{\chi}_0 = D(I_{p_\eta v_1} - \nu_w I_{u_\eta u_\eta 1})/A_t, \tag{63a}$$

$$\hat{\chi}_1 = g_2 - I_{vt_2} - \rho_w^{-1} I_{\rho vt_1} - (\mu\sigma)^{-1} I_{\mu t_\eta 1} - (I_{vt_1}H_{vt_1} + J_{vt_1}^2)A_t/T_w + \Lambda_{t_1}]\sigma/A_t, \tag{63b}$$

$$\hat{\chi}_2 = [J_{uv_2} + \rho_w^{-1}J_{\rho vt_1} - (\mu\sigma)^{-1}J_{\mu t_\eta 1} - A_t J_{vt_1}^2/T_w - A_t(1 + \mu_T T_w/\mu) \times H_{vt_1}J_{vt_1} - DJ_{uv_1}^2]\sigma/A_t, \tag{63c}$$

$$\hat{\chi}_3 = D\sigma(-J_{p_\eta v_1} + \rho_w J_{u_\eta u_\eta 1})/A_t, \tag{63d}$$

$$\hat{\chi}_4 = -(1 + \mu_T T_w/\mu)J_{vt_1}^2\sigma A_t/(2T_w). \tag{63e}$$

The matching condition (47) with $m = n = 3$ for the temperature profile gives

$$\chi_0 = 0, \quad \chi_1 = 0, \quad \chi_3 = 0, \tag{64}$$

and $E_t \ln(E^2R/\nu_w)$ has to be of order unity from (55*b*). Without loss of generality, let

$$[1 + E_t\hat{\chi}_2/J_{vt_1} + \ldots]E_t\ln(E^2R/\nu_w) = \beta_1 + \beta_2 E_t + \beta_3 E_t^2 + O(E_t^2). \tag{65}$$

Introducing (65) in (62) and applying the matching condition with $m = n = 3$ we get

$$\beta_1 J_{vt_1}\sigma + \hat{\chi}_4\beta_1^2 = T_{10} - T_w, \tag{66a}$$

$$h_2(x, Y) \sim (J_{vt_1}\sigma + 2\hat{\chi}_4\beta_1)\ln Y + H_{vt_1}\sigma + 2\hat{\chi}_4\beta_1(\beta_2 - \beta_1\hat{\chi}_2/J_{vt_1}\sigma) + \sigma\beta_2 J_{vt_1} \quad \text{as} \quad Y \to 0, \tag{66b}$$

$$h_3(x, Y) \sim \hat{\chi}_4(\ln Y)^2 + (\hat{\chi}_2 + 2\hat{\chi}_4\beta_2 - 2\hat{\chi}_4\hat{\chi}_2\beta_1/J_{vt_1}\sigma)\ln Y + \beta_3\sigma J_{vt_1} + \Lambda_{t_2} + \hat{\chi}_4\Lambda_{t_3} \quad \text{as} \quad Y \to 0. \tag{66c}$$

The matching of the density distribution leads to expressions similar to those for the velocity and temperature profiles.

## 5. Results and discussion

The inner expansions for the tangential velocity component and temperature are

$$U = E\hat{u}_2(x, \eta) + E^2\hat{u}_3(x, \eta) + O(E^3), \tag{67a}$$

$$U \sim E[J_{uv_1}\ln\eta + H_{uv_1}] + E^2[\chi_4(\ln\eta)^2 + \chi_2\ln\eta + \Lambda_2] + O(E^3) \quad \text{as} \quad \eta \to \infty, \tag{67b}$$

$$T = T_w + E_t\hat{h}_2(x, \eta) + E_t^2\hat{h}_3(x, \eta) + O(E^3), \tag{68a}$$

$$T \sim T_w + E_t[J_{vt_1}\sigma\ln\eta + H_{vt_1}\sigma] + E^2[\hat{\chi}_4(\ln\eta)^2 + \hat{\chi}_2\ln + \Lambda_{t_2}] + O(E^3) \quad \text{as} \quad \eta \to \infty. \tag{68b}$$

The corresponding outer expansions are

$$U = U_{10}(x)+Eu_2(x, Y)+E^2u_3(x, Y)+O(E^3), \tag{69a}$$

$$U \sim U_{10}+E[(J_{uv_1}+2\chi_4\alpha_1)\ln Y - J_{uv_1}\alpha_2+H_{uv_1}+2\chi_4\alpha_1(\alpha_2-\alpha_1\chi_2/J_{uv_1}) + E^2[\chi_4(\ln Y)^2+(\chi_2+2\chi_4\alpha_2-2\alpha_1\chi_2\chi_4/J_{uv_1})\ln Y+J_{uv_1}\alpha_3 + \Lambda_2+\chi_4\Lambda_3]+O(E^3) \quad \text{as} \quad Y\to 0, \tag{69b}$$

$$U \sim U_{10}+EYU_{1y}(x,0)+E^2[U_3(x,0)+\tfrac{1}{2}Y^2U_{1yy}]+O(E^3) \quad \text{as} \quad Y\to\infty, \tag{69c}$$

$$T = T_{10}(x)+E_th_2(x, Y)+E_th_3(x, Y)+O(E_t^3), \tag{70a}$$

$$T \sim T_{10}(x)+E_t[(J_{vt_1}\sigma+2\hat{\chi}_4\beta_1)\ln Y+J_{vt_1}\beta_2\sigma+H_{vt_1}\sigma+2\hat{\chi}_4\beta_1 \times(\beta_2-\beta_1\hat{\chi}_2/J_{vt_1}\sigma)+E_t^2[\hat{\chi}_4(\ln Y)^2+(\hat{\chi}_2+2\hat{\chi}_4\beta_2-2\beta_1\hat{\chi}_4\hat{\chi}_2/J_{vt_1}\sigma) \times\ln Y+\sigma J_{vt_1}\beta_3+\Lambda_{2t}+\hat{\chi}_4\Lambda_{3t}]+O(E_t^3) \quad \text{as} \quad Y\to 0, \tag{70b}$$

$$T \sim T_{10}(x)+E_t\,YT_{1y}(x,0)+E_t^2[T_3(x,0)+\tfrac{1}{2}Y^2T_{1yy}(x,0)]+O(E_t^3) \quad \text{as} \quad Y\to\infty. \tag{70c}$$

The first-order terms (of order $E$) in relations (67$a$) and (68$a$) are the laws of the wall for velocity and temperature distributions. These laws of the wall for a compressible fluid have the same forms as in incompressible flow. The relations (69$a$) and (70$a$) are the corresponding defect laws. The second-order terms of these laws represent the necessary corrections at lower Reynolds number. These expressions show that the effects of entropy gradients and stagnation enthalpy on the outer flow are of the first order, while those of displacement are of second order. In the inner region, the first-order results are independent of viscous dissipation and the second-order terms depend upon mean convection due to the turbulent mass flux, viscous dissipation and work of compressibility. In the overlap region, the first-order results show logarithmic distributions of velocity and temperature. The ratio of the slope of the defect law to that of the law of the wall (later cited as the slope ratio) for the first-order results in the overlap region is $1+2\chi_4\alpha_1J_{uv_1}^{-1}$ for the velocity profile and $1+2\sigma^{-1}\hat{\chi}_4\beta_1J_{vt_1}^{-1}$ for the temperature profile. The second-order terms in the overlap region are the corrections to first-order terms at lower Reynolds number and show bilogarithmic and logarithmic terms. In the above analysis the constants $J_{uv_1}$, $J_{vt_1}$, $\alpha_1$, $\alpha_2$, etc. (in equations (67$b$), (68$b$), (69$b$), (70$b$)) are left unspecified. This feature is a consequence of the underdetermined nature of the system of equations.

Some of the above-mentioned features can also be shown from the work of Rotta (1960), although, in the light of the present work, his analysis is not consistent to order $E^2$ as he has dropped some of the terms (like mean convection $\overline{\rho'v'}U_y$, triple correlations $\overline{\rho'u'v'}$, work of compressibility due to fluctuations $\overline{v'p'_y}$ etc.) of the same order as those retained. Using relations (1$b$) and (2$e$) of the present work and an asymptotic expansion of equation (17) of Rotta (1960) for the law of the wall as $R\to\infty$ for fixed $\eta$, we get

$$U = U_*[k^{-1}\ln\eta+5{\cdot}2]+U_*^2[-0{\cdot}5\sigma_tA_tk^{-1}(\ln\eta)^2+k^{-1}(1{\cdot}7A_t-0{\cdot}1M_\infty(\rho_w/\rho_\infty)^{\frac{1}{2}} -2{\cdot}6\sigma_tA_tk^{-1})\ln\eta+\text{constant}]+\ldots, \quad \text{with} \quad k=0{\cdot}4. \tag{71a}$$

This shows that the first-order results are independent of $M_\infty$ and second-order terms involve bilogarithmic terms. Now in order to show that the slope ratio in the work of Rotta is different from unity, we first note that Rotta has patched his law of the wall to the defect law at the so-called point of maximum temperature, rather than matching them asymptotically. This point of maximum temperature, according to Rotta, is located deep in the sublayer and, therefore, depends upon wall (inner) variables, in particular viscosity. We now write an asymptotic expansion of equation (32) of Rotta for the velocity defect law as $R \rightarrow \infty$ for fixed $y/\Delta$ as

$$\frac{U-U}{U_*^*} = -\frac{1}{k}\ln\left(\frac{y}{\Delta}\right) - \frac{1}{k}\ln\left(\frac{y^*}{\Delta}\right) + \ldots, \quad \text{with} \quad U_*^* = \left(\frac{\tau_w}{\rho^* U_\infty^2}\right)^{\frac{1}{2}}. \tag{71b}$$

Here $U_*^*$ is the friction velocity formed using the density $\rho^*$ at the patching point $y^*$ (given by equation (29) of Rotta) of maximum temperature, which lies in the sublayer and depends upon viscosity. Thus the slope ratio $U_*^*/U_*$ obtained from (71$a$) and (71$b$) is not only, in general, different from unity, but depends upon viscosity.

Further, Baronti & Libby (1966), in their study of point-to-point mapping of compressible turbulent boundary-layer flow into constant-density flow (as suggested by Coles 1964), found that the velocity profiles (with or without heat transfer up a Mach number of 6) in the inner region were well correlated by the law of the wall. However, for the velocity defect law these authors found a systematic degradation with increasing Mach number, or in other words, the additive constant and slope of the defect law in the overlap region can depend upon the Mach number and wall temperature (see figures 5 and 8 of Baronti & Libby 1966). In the present work this slope ratio is found to depend upon the viscosity–temperature relationship and the wall temperature and it does not directly depend upon the Mach number.

The skin friction law as given by (51) is

$$\left(\frac{2}{C_f}\right)^{\frac{1}{2}} = \frac{1}{\alpha_1}\ln\left[\frac{R\Delta}{\nu_w}\left(\frac{C_f}{2}\right)^{\frac{1}{2}}\right] - \frac{\alpha_2}{\alpha_1} + \left(\frac{C_f}{2}\right)^{\frac{1}{2}}\left[\frac{\chi_2}{\alpha_1 J_{uv_1}}\ln\left\{\frac{R\Delta}{\nu_w}\left(\frac{C_f}{2}\right)^{\frac{1}{2}}\right\} - \frac{\alpha_3}{\alpha_1}\right] + O(C_f). \tag{72}$$

In the skin friction law (72) the constant $\alpha_1$ depends upon the viscosity law and wall temperature and can be determined from matching condition (61$a$). The first two terms on the right-hand side have the same form as the classical skin friction law for incompressible flows. Winter & Gaudet (1969) have found, from the analysis of data from various sources, a skin friction correlation similar to that of an incompressible flow which is independent of Mach number up to $M_\infty = 4$. The second-order term in (72) is the necessary correction to the skin friction law at lower Reynolds numbers.

The coefficient of heat transfer $C_h = EE_t$, using (59) and (65), is

$$\frac{1}{C_h} = \frac{\alpha_1}{\beta_1}\left[\frac{2}{C_f} + \phi_1\left(\frac{2}{C_f}\right)^{\frac{1}{2}} + \phi_2 + O(C_f)^{\frac{1}{2}}\right], \tag{73}$$

where $\phi_1$ and $\phi_2$ are given by

$$\phi_1 = [\alpha_2 - \beta_2 + \alpha_1(\chi_2/J_{uv_1} - \hat{\chi}_2 A_t^{-1}/J_{vt_1}\sigma)]/\beta_1, \tag{74a}$$

$$\phi_2 = [\alpha_3 - \beta_3 A_t + \alpha_2(\chi_2/J_{uv_1} - \hat{\chi}_2 A_t^{-1}/J_{vt_1}\sigma)]/\beta_1. \tag{74b}$$

In the heat-transfer law, the first-order terms depend upon the wall temperature and viscosity–temperature relationship, and the second-order terms depend upon $M_\infty$.

Uniformly valid expansions can be obtained by taking the unions of inviscid, outer and inner expansions and then subtracting the common parts (see Van Dyke 1964).

The first-order analysis of the present work shows that the effects of high Mach number are mainly due to variable fluid properties, rather than viscous dissipation. This suggests that, experimentally, we can study the structure of a compressible turbulent boundary layer at *low* Mach number but with variable fluid properties. A similar suggestion was made by Morkovin (1964).

The analysis of incompressible flow carried out in the appendix exhibits logarithmic distributions in the overlap region to order $(E^3R)^{-1}$, while for compressible flow only the first-order terms show a logarithmic distribution, and the second-order show bilogarithmic terms.

The present analysis does not include other effects like curvature and low density (non-continuum). The non-continuum effects of velocity slip and temperature jump can be studied in the framework of the Reynolds formulation. For these effects the reader may refer to Afzal (1971). Analysis of the problem dealing with effects of curvature is currently in progress and author hopes to publish the results in near future.

It is a pleasure to thank Prof. Roddam Narasimha for many helpful discussions. I have also been benefited from discussion with Prof. M. R. Head, K. Yajnik, A. Prabhu, M. M. Oberai and D. B. Spalding. I am also thankful to the referees for some very helpful comments.

## Appendix. Analysis for incompressible flow

Here we shall describe the effect of pressure gradients [of order $(E^3R)^{-1}$] on the law of the wall and the velocity defect law. To evaluate this effect we need to calculate many higher order terms and thus the analysis for a compressible flow becomes very complicated. Therefore, we shall study the effects of pressure gradient for an incompressible flow only. The governing equations for an incompressible flow are

$$U_x + V_y = 0, \tag{A 1}$$

$$UU_x + VU_y + P_x = R^{-1}(U_{xx} + U_{yy}) + \tau_{uux} + \tau_{uvy}, \tag{A 2}$$

$$UV_x + VV_y + P_y = R^{-1}(V_{xx} + V_{yy}) + \tau_{uvx} + \tau_{vvy}. \tag{A 3}$$

The boundary conditions at the wall are

$$U(x,0) = 0, \quad V(x,0) = 0.$$

(*a*) *Inviscid layer.* In order to study the problem to order $(E^3R)^{-1}$ we have to consider double expansions of the type described by equation (17*b*). Thus we assume the following expansion:

$$U = \sum_{m=1}^{\infty} U_m(x,y)\, E^{m-1} + (E^2R)^{-1}\, \tilde{U}_1(x,y) + O(ER)^{-1}, \tag{A 4}$$

and similar expansions for the other variables. The problems governing $U_n$, $\tilde{U}_1$, etc. can immediately be written down from the governing equations (A 1)–(A 3). It turns out that the equations governing $U_1, U_2, \ldots$, and $\tilde{U}_1, \ldots$, are inviscid. As such, the inviscid expansion cannot, in general, satisfy the no-slip condition.

(*b*) *Outer layer*. For the outer layer, the asymptotic expansions are

$$U = \sum_{m=1}^{\infty} u_m(x, Y)\,E^{m-1} + (E^2R)^{-1}\tilde{u}_1(x, Y) + O(ER)^{-1}, \tag{A 5a}$$

$$V = \sum_{m=1}^{\infty} v_m(x, Y)\,E^{m} + (ER)^{-1}\tilde{v}_1(x, Y) + O(R^{-1}), \tag{A 5b}$$

$$P = \sum_{m=1}^{\infty} p_m(x, Y)\,E^{m-1} + (E^2R)^{-1}\tilde{p}_1(x, Y) + O(ER)^{-1}, \tag{A 5c}$$

$$\tau_{ab} = \sum_{m=1}^{\infty} \tau_{ab_m}(x, Y)\,E^{m+1} + R^{-1}\tilde{\tau}_{ab_1}(x, Y) + O(ER^{-1}). \tag{A 5d}$$

The problems governing $u_n$ etc. are inviscid in the sense that these do not involve the viscous terms, while the problem governing $\tilde{u}_1$ involves the viscous term $u_{1YY}$. However, this outer layer also cannot, in general, satisfy the no-slip condition at the wall.

The matching of the inviscid expansions and the outer expansions can be carried out in a manner similar to that described in §4.3.

(*c*) *Inner layer*. In the inner layer the appropriate asymptotic expansions are

$$U = \sum_{m=1}^{\infty} \hat{u}_m(x, \eta)\,E^{m} + (E^2R)^{-1}\hat{\hat{u}}_1(x, \eta) + O(ER)^{-1}, \tag{A 6a}$$

$$p = \hat{p}_1(x, \eta) + E\hat{p}_2(x, \eta) + E^2\hat{p}_3(x, \eta) + O(E^3), \tag{A 6b}$$

$$\tau_{ab} = \sum_{m=1}^{\infty} \hat{\tau}_{ab_m}(x, \eta)\,E^{m+1} + R^{-1}\hat{\hat{\tau}}_{ab_1}(x, \eta) + O(ER^{-1}). \tag{A 6c}$$

The expansion (A 6*a*) shows that the $V$ component of velocity in the inner region is of the order of $R^{-1}$, while for the compressible flow, this velocity was shown to be of order $E^2$. The equations governing the inner layer are

$$\hat{u}_{m\eta\eta} + \hat{\tau}_{uv_m\eta} = 0 \quad (m = 1, 2, 3, \ldots), \tag{A 7a}$$

$$\hat{\hat{u}}_{1\eta\eta} + \hat{\hat{\tau}}_{uv_1\eta} = p_{1x} = -U_{10}U_{10x}. \tag{A 7b}$$

We now consider the matching of the outer and inner expansions. For Reynolds stresses, the inner limit of the outer expansion (A 5*d*) is

$$\tau_{uv} \sim \sum_{m=1}^{\infty} \tau_{uv_m}(x, 0)\,E^{m+1} + (E^2R)^{-1}[\tilde{\tau}_{uv_1}(x, 0) + \eta\tau_{uv_1Y}(x, 0)] + O(ER)^{-1} \quad \text{as} \quad \eta \to \infty. \tag{A 8}$$

Matching of (A 8) with the outer limit of the inner expansion (A 6*d*) leads to

$$\hat{\tau}_{uv_m}(x, \eta) \sim \tau_{uv_m}(x, 0) - J_{uv_m}/\eta + \ldots \quad \text{as} \quad \eta \to \infty, \tag{A 9}$$

$$\tilde{\tau}_{uv_1}(x, \eta) \sim \eta\tau_{uv_1Y}(x, 0) + \tilde{\tau}_{uv_1}(x, 0) - \tilde{J}_{uv_1}/\eta + \ldots \quad \text{as} \quad \eta \to \infty. \tag{A 10}$$

On using (A 9) and (A 10), the integration of equations (A 7*a*) and (A 7*b*) gives

$$\hat{u}_m = [f_m - \tau_{uv_m}(x,0)]\,\eta + J_{uv_m}\ln(1+\eta) + H_{uv_m}, \tag{A 11a}$$

$$\hat{\tilde{u}}_1 = -\tfrac{1}{2}[U_{10}U_{10x} + \tau_{uv_1Y}(x,0)]\,\eta^2 + [\tilde{f}_1 - \tilde{\tau}_{uv_1}(x,0)]\,\eta + \tilde{J}_{uv_1}\ln(1+\eta) + \tilde{H}_{uv_1}. \tag{A 11b}$$

The outer limit of the inner expansion (A 6*a*) may be written using (A 11) as

$$\begin{aligned} U = {} & \sum_{m=1}^{\infty} E^m[\{f_m - \tau_{uv_m}(x,0)\}\,YE^2R + J_{uv_m}\ln(E^2R) + J_{uv_m}\ln Y + H_{uv_m}] \\ & -\tfrac{1}{2}[U_{10}U_{10x} + \tau_{uv_1Y}(x,0)]\,Y^2R^2 + [\tilde{f}_1 - \tilde{\tau}_{uv_1}(x,0)]\,Y + (E^2R)^{-1} \\ & \times[\tilde{J}_{uv_1}\ln Y + \tilde{J}_{uv_1}\ln(E^2R) + \tilde{H}_{uv_1}] + \dots . \end{aligned} \tag{A 12}$$

Now the inner limit of the outer expansion (A 5*a*) shows that matching with (A 13) leads to

$$f_m = \tau_{uv_m}(x,0), \quad \tau_{uv_1Y}(x,0) = -U_{10}U_{10x}, \tag{A 13}$$

and $E\ln(E^2R)$ has to be of order unity. Without loss of generality let

$$\left[\sum_{m=1}^{\infty} J_{uv_m}E^{m-1} + (E^3R)^{-1}\tilde{J}_{uv_1} + \dots\right] E\ln(E^2R) = \sum_{m=1}^{\infty} B_{m-1}E^{m-1} + (E^3R)^{-1}\tilde{B}_1 + \dots . \tag{A 14}$$

By introducing (A 14) in (A 12) and matching it with the inner limit of the outer solution we get

$$\left.\begin{aligned} & u_1(x,Y) \sim B_0 + [\tilde{f}_1 - \tilde{\tau}_{uv_1}(x,0)]\,Y \quad \text{as} \quad Y \to 0, \\ & u_m(x,Y) \sim J_{uv_m}\ln Y + H_{uv_m} + B_m \quad (m = 1,2,3,\dots) \quad \text{as} \quad Y \to 0, \\ & \tilde{u}_1(x,Y) \sim \tilde{J}_{uv_1}\ln Y + \tilde{H}_{uv_1} + \tilde{B}_1 \quad \text{as} \quad Y \to 0. \end{aligned}\right\} \tag{A 15}$$

(*d*) *Results.* In the overlap region the law of the wall and the velocity defect law are

$$U \sim \sum_{m=1}^{\infty} E^m[J_{uv_m}\ln\eta + H_{uv_m}] + (E^2R)^{-1}[\tilde{J}_{uv_1}\ln\eta + \tilde{H}_{uv_1}] + O(ER)^{-1} \quad \text{as} \quad \eta \to \infty, \tag{A 16}$$

$$U \sim U_{10} + \sum_{m=1}^{\infty} E^m[J_{uv_m}\ln Y + B_m + H_{uv_m}] + (E^2R)^{-1} \times[\tilde{J}_{uv_1}\ln Y + \tilde{H}_{uv_1} + \tilde{B}_1] + O(ER)^{-1} \quad \text{as} \quad Y \to 0. \tag{A 17}$$

The skin friction law as given by (A 14) is

$$\begin{aligned} \left(\frac{2}{C_{f0}}\right)^{\frac{1}{2}} = \frac{1}{B_0}\Bigg[ & \left\{\Sigma_1 J_{uv_m}\left(\frac{C_{f0}}{2}\right)^{\frac{1}{2}(m-1)} + R^{-1}\left(\frac{C_{f0}}{2}\right)^{-\frac{3}{2}}\tilde{J}_{uv_1} + \dots\right\}\ln\left[R\Delta\left(\frac{C_{f0}}{2}\right)^{\frac{1}{2}}\right] \\ & - \Sigma_1 B_m\left(\frac{C_{f0}}{2}\right)^{\frac{1}{2}(m-1)} + R^{-1}\left(\frac{C_{f0}}{2}\right)^{-\frac{3}{2}}\tilde{B}_1 + O(RC_{f0})^{-1}\Bigg]. \end{aligned} \tag{A 18}$$

In relations (A 16)–(A 18) the first-order terms are similar to those of Millikan (1939) and Yajnik (1970). Further, these relations show that the incompressible turbulent boundary layer has logarithmic terms to order $(E^3R)^{-1}$ in contrast to compressible flow, when logarithmic behaviour is observed for lowest order terms.

The results (A 16)–(A 18) can be written in the classic well-known forms

$$U \sim \begin{cases} E[A\ln\eta + H] & \text{as} \quad \eta \to \infty, \quad \text{(A 19)} \\ U_{10} + E[A\ln Y + B] & \text{as} \quad Y \to 0, \quad \text{(A 20)} \end{cases}$$

$$(2/C_{f0})^{\frac{1}{2}} = A\ln[R\Delta(\tfrac{1}{2}C_{f0})^{\frac{1}{2}}] + H - B, \quad \text{(A 21)}$$

where

$$\left.\begin{aligned} A &= \sum_{m=1}^{\infty} E^{m-1}J_{uv_m} + (E^3R)^{-1}\tilde{J}_{uv_1} + \ldots, \\ H &= \sum_{m=1}^{\infty} E^{m-1}H_{uv_m} + (E^3R)^{-1}\tilde{H}_{uv_1} + \ldots, \\ B &= \sum_{m=1}^{\infty} E^{m-1}(B_m + H_{uv_m}) + (E^3R)^{-1}(\tilde{B}_1 + \tilde{H}_{uv_1}) + \ldots. \end{aligned}\right\} \quad \text{(A 22)}$$

These results show that the classical forms of the law of wall (A 19), velocity defect law (A 20) and skin friction law (A 21) can describe the effects of lower Reynolds number provided that the constants in these classical laws are treated as functions of Reynolds number according to the above-mentioned expressions (A 22). In these expressions the last term shows that the effects of pressure gradient on the turbulent boundary layer are of order $(E^3R)^{-1}$. In the above analysis the constants $J_{uv_m}$, $H_{uv_m}$, $B_m$, etc., are left unspecified. This is because the present work deals with an underdetermined set of equations of mean motion, i.e. without a set of closure hypotheses.

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